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Periodic-like solutions of variable coefficient and Wick-type stochastic NLS equations ☆

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Abstract

Variable coefficient and Wick-type stochastic nonlinear Schrödinger (NLS) equations are investigated. By using white noise analysis, Hermite transform and extended F-expansion method, we obtain a number of Wick versions of periodic-like wave solutions and periodic wave solutions expressed by various Jacobi elliptic functions for Wick-type stochastic and variable coefficient NLS equations, respectively. In the limit cases, the soliton-like wave solutions are showed as well. Since Wick versions of functions are usually difficult to evaluate, we get some nonWick versions of the solutions for Wick-type stochastic NLS equations in special cases.
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1. Introduction

It is well known that the nonlinear Schrödinger (NLS) equations play an essential role for the understanding of many physical phenomena: wave propagation in nonlinear media, fluid mechanics, quantum mechanics, plasma physics, etc. The numerical solutions of Schrödinger equations have researched by many authors, they are Tascli [17], Konguetsof and Simos [11], Kalogiratou et al. [10] and Sakas and Simos [15], etc.

Variable coefficient NLS equations are the ones of the most fundamental equations of quantum mechanics, which describe certain physical systems with nonuniform backgrounds and have many physical applications, for example, in gravitation, movement of trapped ions, coherent state studies and so on. In this paper, we will study the following variable coefficient NLS equation:

$$\mathrm{i}u_t + f(t)u_{xx} + g(t)u|u|^2 = 0, \tag{1.1}$$

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where $f(t)$ and $g(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$ with $f(t)g(t) \neq 0$. There are many authors to study Eq. (1.1), e.g., Ablowitz and Clarkson [1], Gao and Tian [8,18] and many others. If the problem is considered in random environment, we can get random NLS equations. In order to give the exact solutions, we only consider the Wick versions in white noise environment, namely, we only consider the following Wick-type stochastic NLS equation:

$$\mathrm{i}U_t + F(t) \diamond U_{xx} + G(t) \diamond U \diamond |U|^{\diamond 2} = 0, \tag{1.2}$$

where "$\diamond$" is the Wick product on the Kondratiev distribution space $(\mathscr{S})_{-1}$ which is defined in the second section, $F(t)$ and $G(t)$ are $(\mathscr{S})_{-1}$-valued functions with $F(t) \diamond G(t) \neq 0$.

Random waves are an important subject of random PDEs. There are many authors to study them. As far as we know that Wadati first introduced and studied the stochastic KdV equation and gave the diffusion of soliton of the KdV equation under Gaussian noise in [19], Xie firstly researched Wick-type stochastic KdV equation on white noise space and showed the auto-Bäcklund transformation and the exact white noise functional solutions in [22], furthermore, Chen [3–6] and Xie [23–28] researched some Wick-type stochastic wave equations using white noise analysis method. Random NLS equations were studied in [12,7,14,16]. Recently, Wang and Li [20] gave the extended F-expansion method, Lü and Zhang [13] showed a further extended tanh method, these methods has been applied to derive nonlinear transformations and exact solutions of nonlinear PDEs in mathematical physics. In this paper, we will use white noise analysis, Hermite transform and extended F-expansion method to get periodic wave solutions and Wick versions of periodic-like wave solutions expressed by various Jacobi elliptic functions for Eqs. (1.1) and (1.2), respectively. The soliton-like wave solutions are showed as well in the limit cases. Since Wick versions of functions are usually difficult to evaluate, we also get some nonWick versions of the solutions for Wick-type stochastic NLS equations in special cases.

2. SPDEs driven by white noise

In this section the main matters for stochastic partial differential equations with white noise functional approach is to be summarized. Please see Holden et al.'s book [9] for detail.

Let $h_n(x)$ be the Hermite polynomials. Put $\xi_n(x) = \mathrm{e}^{-(1/2)x^2} h_n(\sqrt{2}x)/(\pi(n-1)!)^{1/2}$, $n \geqslant 1$. We have that the collection $\{\xi_n\}_{n\geqslant 1}$ constitutes an orthogonal basis for $L^2(\mathbb{R})$.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be d-dimensional multi-indices with $\alpha_1, \ldots, \alpha_d \in \mathbb{N}$, then the family of tensor products $\xi_\alpha = \xi_{(\alpha_1,\ldots,\alpha_d)} = \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_d}$ $(\alpha \in \mathbb{N}^d)$ forms an orthogonal basis for $L^2(\mathbb{R}^d)$. And let $\alpha^{(i)} = (\alpha_1^{(i)}, \ldots, \alpha_d^{(i)})$ be the ith multi-index number in some fixed ordering of all d-dimensional multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. We can, and will, assume that this ordering has the property

$$i < j \Rightarrow \alpha_1^{(i)} + \cdots + \alpha_d^{(i)} \leqslant \alpha_1^{(j)} + \cdots + \alpha_d^{(j)}.$$

Now define

$$\eta_i = \xi_{\alpha^{(i)}} = \xi_{\alpha_1^{(i)}} \otimes \cdots \otimes \xi_{\alpha_d^{(i)}}, \quad i \geqslant 1.$$

Put $(\mathbb{N}_0^{\mathbb{N}})_c = \{\alpha = (\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{N}_0, \alpha_i = 0, i \geqslant n \text{ for some } n \geqslant 1\}$. For $\alpha = (\alpha_1, \alpha_2, \ldots) \in (\mathbb{N}_0^{\mathbb{N}})_c$, we define

$$H_\alpha(\omega) = \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \eta_i \rangle), \quad \omega \in (S(\mathbb{R}^d))^*.$$

Fixed $n \in \mathbb{N}$, let $(\mathscr{S})_1^n$ consist of those $x = \sum_\alpha c_\alpha H_\alpha \in \bigoplus_{k=1}^n L^2(\mu)$ with $c_\alpha \in \mathbb{R}^n$ such that $\|x\|_{1,k}^2 = \sum_\alpha c_\alpha^2 (\alpha!)^2 (2\mathbb{N})^{k\alpha} < \infty$, $\forall k \in \mathbb{N}$ with $c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^n (c_\alpha^{(k)})^2$ if $c_\alpha = (c_\alpha^{(1)}, \ldots, c_\alpha^{(n)}) \in \mathbb{R}^n$, where μ is the white noise measure on $(\mathscr{S}^*(\mathbb{R}), \mathscr{B}(\mathscr{S}^*(\mathbb{R})))$, $\alpha = \prod_{k=1}^\infty \alpha_k!$ and $(2\mathbb{N})^\alpha = \prod_j (2j)^{\alpha_j}$ for $\alpha = (\alpha_1, \alpha_2, \ldots) \in (\mathbb{N}_0^{\mathbb{N}})_c$, $(\mathscr{S}(\mathbb{R}^d))$ and $(\mathscr{S}(\mathbb{R}^d))^*$ are the Hida test function space and the Hida distribution space on $\mathbb{R}^d$, respectively.

The space $(\mathscr{S})_{-1}^n$ consists of all formal expansions $X = \sum_\alpha b_\alpha H_\alpha$ with $b_\alpha \in \mathbb{R}^n$ such that $\|X\|_{-1,-q} = \sum_\alpha b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$ for some $q \in \mathbb{N}$. The family of seminorms $\|x\|_{1,k}$, $k \in \mathbb{N}$ gives rise to a topology on $(\mathscr{S})_1^n$, and we can regard $(\mathscr{S})^n_{-1}$ as the dual of $(\mathscr{S})^n_1$ by the action

$$\langle X, x\rangle = \sum_\alpha (b_\alpha, c_\alpha)\alpha!$$

and (b_α, c_α) is the usual inner product in $\mathbb{R}^n$.

For $X = \sum_\alpha a_\alpha H_\alpha,\ Y = \sum_\alpha b_\alpha H_\alpha \in (\mathscr{S})^n_{-1}$ with $a_\alpha,\ b_\alpha \in \mathbb{R}^n$,

$$X \diamond Y = \sum_{\alpha,\beta} (a_\alpha, b_\beta) H_{\alpha+\beta}$$

is called the Wick product of X and Y.

We can prove that the spaces $(\mathscr{S}(\mathbb{R}^d))$, $(\mathscr{S}(\mathbb{R}^d))^*$, $(\mathscr{S})_1$ and $(\mathscr{S})_{-1}$ are closed under Wick products.

For $X = \sum_\alpha a_\alpha H_\alpha \in (\mathscr{S})^n_{-1}$ with $a_\alpha \in \mathbb{R}^n$, the Hermite transform of X, denoted by $\mathscr{H}(X)$ or $\widetilde{X}$, is defined by

$$\mathscr{H}(X) = \widetilde{X}(z) = \sum_\alpha a_\alpha z^\alpha \in \mathbb{C}^n \quad \text{(when convergent)},$$

where $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ (the set of all sequences of complex numbers) and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \cdots$ for $\alpha = (\alpha_1, \alpha_2, \ldots) \in (\mathbb{N}_0^{\mathbb{N}})_c$.

For $X, Y \in (\mathscr{S})^n_{-1}$, by this definition we have

$$\widetilde{X \diamond Y}(z) = \widetilde{X}(z) \cdot \widetilde{Y}(z)$$

for all z such that $\widetilde{X}(z)$ and $\widetilde{Y}(z)$ exist. The product on the right-hand side of the above formula is the complex bilinear product between two elements of $\mathbb{C}^{\mathbb{N}}$ defined by $(z_1^1, \ldots, z_n^1) \cdot (z_1^2, \ldots, z_n^2) = \sum_{k=1}^n z_k^1 z_k^2$, where $z_k^i \in \mathbb{C}$.

Let $X = \sum_\alpha a_\alpha H_\alpha \in (\mathscr{S})^n_{-1}$. Then the vector $c_0 = \widetilde{X}(0) \in \mathbb{R}^n$ is called the generalized expectation of X and denoted by $E(X)$. Suppose that $f : V \to \mathbb{C}^m$ is an analytic function, where V is a neighborhood of $E(X)$. Assume that the Taylor series of f around $E(X)$ has coefficients in $\mathbb{R}^n$, then the Wick version $f^\diamond(X) = \mathscr{H}^{-1}(f \circ \widetilde{X}) \in (S)^m_{-1}$.

The Wick exponential of $X \in (\mathscr{S})_{-1}$ is defined by $\exp^\diamond\{X\} = \sum_{n=0}^\infty X^{\diamond n}/n!$. With the Hermite transform the Wick exponential shows the same algebraic properties as the usual one. For example, $\exp^\diamond\{X + Y\} = \exp^\diamond\{X\} \diamond \exp^\diamond\{Y\}$.

The modelling consideration leads us to think an SPDE expressed formally as $A(t, x, \partial_t, \nabla_x, U, \omega) = 0$, where A is a given function, $U = U(t, x, \omega)$ is the unknown (generalized) stochastic process, and the operators $\partial_t = \partial/\partial t$, $\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$ when $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. First we interpret all products as Wick products and all functions as their Wick versions. We indicate this as

$$A^\diamond(t, x, \partial_t, \nabla_x, U, \omega) = 0. \tag{2.1}$$

Second, we take the Hermite transform of (2.1). This turns Wick products into ordinary products (between complex numbers) and the equation takes the form

$$\widetilde{A}(t, x, \partial_t, \nabla_x, \widetilde{U}, z_1, z_2, \ldots) = 0, \tag{2.2}$$

where $\widetilde{U} = \mathscr{H}(U)$ is the Hermite transform of U and $z_1, z_2, \ldots$ are complex numbers. Suppose we can find a solution $u = u(t, x, z)$ of the equation $\widetilde{A}(t, x, \partial_t, \nabla_x, u, z) = 0$ for each $z = (z_1, z_2, \ldots) \in \mathbb{K}_q(r)$ for some $q,\ r$, where $\mathbb{K}_q(r) = \{z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ and $\sum_{\alpha \neq 0} |z^\alpha|^2 (2\mathbb{N})^{q\alpha} < r^2\}$. Then, under certain conditions we can take the inverse Hermite transform $U = \mathscr{H}^{-1} u \in (\mathscr{S})_{-1}$ and thereby obtain a solution U of the original Wick equation (2.1). We have the following theorem, which was proved in [9].

Theorem 2.1. *Suppose $u(t, x, z)$ is a solution (in the usual strong, pointwise sense) of Eq.* (2.2) *for (t, x) in some bounded open set $\mathbf{G} \subset \mathbb{R} \times \mathbb{R}^d$, and for all $z \in \mathbb{K}_q(r)$, for some $q,\ r$. Moreover, suppose that $u(t, x, z)$ and all its partial derivatives, which are involved in* (2.2), *are bounded for $(t, x, z) \in \mathbf{G} \times \mathbb{K}_q(r)$, continuous with respect to $(t, x) \in \mathbf{G}$ for all $z \in \mathbb{K}_q(r)$ and analytic with respect to $z \in \mathbb{K}_q(r)$, for all $(t, x) \in \mathbf{G}$.*

Then there exists $U(t, x) \in (\mathscr{S})_{-1}$ such that $u(t, x, z) = (\widetilde{U}(t, x))(z)$ for all $(t, x, z) \in \mathbf{G} \times \mathbb{K}_q(r)$ and $U(t, x)$ solves (in the strong sense in $(\mathscr{S})_{-1}$) Eq. (2.1) *in $(\mathscr{S})_{-1}$.*

3. Exact solutions of (1.1) and (1.2)

In this section, we will use Theorem 2.1 with $d=1$ to obtain white noise functional solutions of (1.2) and exact solutions of (1.1).

Taking the Hermite transform of Eq. (1.2), we can get the next equation

$$\mathrm{i}\widetilde{U}_t(t,x,z)+\widetilde{F}(t,z)\widetilde{U}_{xx}(t,x,z)+\widetilde{G}(t,z)\widetilde{U}(t,x,z)|\widetilde{U}(t,x,z)|^2=0, \tag{3.1}$$

where $z=(z_1,z_2,\ldots)\in\mathbb{C}^{\mathbb{N}}$ is a parameter. The condition $F(t)\diamond G(t)\neq 0$ yields $\widetilde{F}(t,z)\widetilde{G}(t,z)\neq 0$. Now, we first solve Eq. (3.1).

Let $u(t,x,z)=\widetilde{U}(t,x,z)$, $f(t,z)=\widetilde{F}(t,z)$, $g(t,z)=\widetilde{G}(t,z)$, we assume that the solutions of (3.1) have the form

$$u(t,x,z)=\exp\{\mathrm{i}[\alpha(t,z)x+\beta(t,z)]\}w(\mu(t,z)x+v(t,z)), \tag{3.2}$$

where $w(\xi)$, $\alpha(t,z)$, $\beta(t,z)$ $\mu(t,z)$ and $v(t,z)$ will be determined later. Put $\eta(t,z)=\alpha(t,z)x+\beta(t,z)$, $\xi(t,z)=\mu(t,z)x+v(t,z)$. Substituting (3.2) into (3.1), we have

$$\begin{aligned}&\mathrm{i}w_t(t,x,z)+f(t,z)w_{xx}(t,x,z)+g(t,z)w(t,x,z)|w(t,x,z)|^2\\&\quad=[\mathrm{i}\mu_t(t,z)w'(\xi)-\alpha_t(t,z)w(\xi)]x+\mathrm{i}[v_t(t,z)+2\alpha(t,z)\mu(t,z)f(t,z)]w(\xi)\\&\qquad-[\beta_t(t,z)+\alpha^2(t,z)f(t,z)]w(\xi)+\mu^2(t,z)f(t,z)w''(\xi)+g(t,z)w^3(\xi)=0.\end{aligned} \tag{3.3}$$

This implies

$$\mathrm{i}\mu_t(t,z)w'(\xi)-\alpha_t(t,z)w(\xi)=0, \tag{3.4}$$

$$[v_t(t,z)+2\alpha(t,z)\mu(t,z)f(t,z)]w(\xi)=0 \tag{3.5}$$

and

$$-[\beta_t(t,z)+\alpha^2(t,z)f(t,z)]w(\xi)+\mu^2(t,z)f(t,z)w''(\xi)+g(t,z)w^3(\xi)=0. \tag{3.6}$$

Eq. (3.4) means $\alpha_t(t,z)=\mu_t(t,z)=0$. These imply that $\alpha(t,z)$ and $\mu(t,z)$ are constants, denoted them by α and μ, respectively. From (3.5), we have $v_t(t,z)=-2\alpha\mu f(t,z)$, this means

$$v(t,z)=-2\alpha\mu\int_0^t f(s,z)\,\mathrm{d}s. \tag{3.7}$$

Suppose $\beta_t(t,z)=C_1f(t,z)$, $g(t,z)=C_2f(t,z)$ with $C_i\neq 0$ $(i=1,2)$, we obtain

$$\beta(t,z)=C_1\int_0^t f(s,z)\,\mathrm{d}s \tag{3.8}$$

and an ODE

$$\mu^2w''(\xi)-[C_1+\alpha^2]w(\xi)+C_2w^3(\xi)=0. \tag{3.9}$$

In the following, we will solve the ODE (3.9).

Considering the homogeneous balance between w'' and w^3 in (3.9), we assume that $w(\xi)$ can be expressed by the extended *F*-expansion in the form

$$w(\xi)=a_1\varphi^{-1}(\xi)+a_2\varphi(\xi)+a_3\psi(\xi), \tag{3.10}$$

where $\varphi(\xi)$ and $\psi(\xi)$ satisfy the following ODEs

$$(\varphi')^2=b_1\varphi^4+b_2\varphi^2+b_3, \tag{3.11}$$

and

$$(\psi')^2=c_1\psi^4+c_2\psi^2+c_3, \tag{3.12}$$

respectively, and

$$\psi^2 = \gamma\varphi^2 + \delta, \tag{3.13}$$

where a_i $(a_2 \neq 0)$ $(i = 1, 2, 3)$ are constants to be determined later, b_i, c_i $(i = 1, 2, 3)$ and γ, δ are constants to be selected such that the solutions of (3.11) and (3.12) are different Jacobi elliptic functions satisfying relation (3.13). It is easy to prove

$$b_1 = \gamma c_1, \quad 3\delta c_1 = b_2 - c_2. \tag{3.14}$$

Substituting (3.10) with (3.11)–(3.13) into (3.9) and using linear independence of $\varphi^i\psi^j$, $(i = 0, \ldots, 6,\ j = 0, 1)$, we have the system of equations:

$$\begin{aligned}
&a_1^2 a_3 C_2 = 0,\\
&2\mu^2 a_1 b_3 + a_1^3 C_2 = 0,\\
&-\alpha^2 a_1 + \mu^2 a_1 b_2 - a_1 C_1 + 3a_1^2 a_2 C_2 + 3\delta a_1 a_3^2 C_2 = 0,\\
&-\alpha^2 a_2 + \mu^2 a_2 b_2 - a_2 C_1 + 3a_1 a_2^2 C_2 + 3\gamma a_1 a_3^2 C_2 + 3\delta a_2 a_3^2 C_2 = 0,\\
&2\mu^2 a_2 b_1 + a_2^3 C_2 + 3\gamma a_2 a_3^2 C_2 = 0,\\
&-\alpha^2 a_3 + 2\delta\mu^2 a_3 c_1 + \mu^2 a_3 c_2 - a_3 C_1 + 6a_1 a_2 a_3 C_2 + \delta a_3^3 C_2 = 0,\\
&2\gamma\mu^2 a_3 c_1 + 3a_2^2 a_3 C_2 + \gamma a_3^3 C_2 = 0.
\end{aligned} \tag{3.15}$$

For these equations, we have the following three kinds of solutions:
(I) (a) For $a_1 = 0$, $a_3 \neq 0$ and $\delta \neq 0$, using (3.14) we have

$$\begin{aligned}
&\alpha = \pm\sqrt{\frac{\mu^2(b_2 + c_2) - 2C_1}{2}}, \quad a_2 = \pm\mu\sqrt{\frac{-\gamma c_1}{2C_2}},\\
&a_3 = \pm\mu\sqrt{\frac{-c_1}{2C_2}}, \quad b_1 = \gamma c_1.
\end{aligned} \tag{3.16}$$

(b) For $a_1 = 0$, $a_3 \neq 0$ and $\gamma \neq 0$,

$$\begin{aligned}
&\alpha = \pm\sqrt{\mu^2 b_2 - C_1}, \quad a_2 = \pm\mu\sqrt{\frac{-\gamma c_1}{2C_2}},\\
&a_3 = \pm\mu\sqrt{\frac{-c_1}{2C_2}}, \quad b_2 = c_2.
\end{aligned} \tag{3.17}$$

(II) For $a_3 = 0$, $a_1 a_2 \neq 0$ and $b_1 \neq 0$,

$$\begin{aligned}
&\alpha = \pm\sqrt{-\sqrt{2}C_1 - \frac{a_2^2 b_2 C_2}{\sqrt{2b_1}} - \frac{6a_2^2\sqrt{b_3}C_2}{\sqrt{2b_1}}},\\
&\mu = \pm \mathrm{i} a_2\sqrt{\frac{C_2}{2b_1}}, \quad a_1 = \pm a_2\sqrt{\frac{b_3}{b_1}}.
\end{aligned} \tag{3.18}$$

(III) For $a_1 = a_3 = 0$ and $b_1 \neq 0$,

$$\alpha = \pm\sqrt{-C_1 - \frac{a_2^2 b_2 C_2}{2b_1}}, \quad \mu = \pm \mathrm{i} a_2\sqrt{\frac{C_2}{2b_1}}. \tag{3.19}$$

Remark. There are eight possible combinations of "+" and "−" in (3.16)–(3.18), and there are four possible combinations of "+" and "−" in (3.19).

Substituting (3.10) with (3.7), (3.8) and (3.16)–(3.19) into (3.2), we get three kinds of concentration formulas of solutions of (3.1) as follows with choosing $\varepsilon_i = 1$ or -1 $(i = 1, 2, 3)$.

(I) (a) For $a_1 = 0$, $a_3 \neq 0$ and $\delta \neq 0$, we have the following eight possible solutions:

$$\begin{aligned} u(t,x,z) = \mu \exp\left\{ \mathrm{i}\left[\varepsilon_2 x \sqrt{\frac{\mu^2(b_2+c_2)-2C_1}{2}} + C_1 \int_0^t f(s,z)\,\mathrm{d}s \right]\right\} \\ \times \left\{ \varepsilon_1 \sqrt{\frac{-\gamma c_1}{2C_2}} \varphi\left(\mu x - 2\varepsilon_2 \mu \sqrt{\frac{\mu^2(b_2+c_2)-2C_1}{2}} \int_0^t f(s,z)\,\mathrm{d}s \right) \right. \\ \left. + \varepsilon_3 \sqrt{\frac{-c_1}{2C_2}} \psi\left(\mu x - 2\varepsilon_2 \mu \sqrt{\frac{\mu^2(b_2+c_2)-2C_1}{2}} \int_0^t f(s,z)\,\mathrm{d}s \right) \right\}. \end{aligned} \tag{3.20}$$

(b) For $a_1 = 0$, $a_3 \neq 0$ and $\gamma \neq 0$, we have the following eight possible solutions with $b_2 = c_2$:

$$\begin{aligned} u(t,x,z) = \mu \exp\left\{ \mathrm{i}\left(\varepsilon_2 x \sqrt{\mu^2 b_2 - C_1} + C_1 \int_0^t f(s,z)\,\mathrm{d}s \right)\right\} \\ \times \left\{ \varepsilon_1 \sqrt{\frac{-\gamma c_1}{2C_2}}\, \varphi\left(\mu x - 2\varepsilon_2 \mu \sqrt{\mu^2 b_2 - C_1} \int_0^t f(s,z)\,\mathrm{d}s \right) \right. \\ \left. + \varepsilon_3 \sqrt{\frac{-c_1}{2C_2}}\, \psi\left(\mu x - 2\varepsilon_2 \mu \sqrt{\mu^2 b_2 - C_1} \int_0^t f(s,z)\,\mathrm{d}s \right) \right\}. \end{aligned} \tag{3.21}$$

(II) For $a_3 = 0$, $a_1 a_2 \neq 0$ and $b_1 \neq 0$, we have the following eight possible solutions with $b_2 = c_2$:

$$\begin{aligned} u(t,x,z) = a_2 \exp\left\{ \mathrm{i}\left(\varepsilon_1 x \sqrt{-\sqrt{2}C_1 - \frac{a_2^2 b_2 C_2}{\sqrt{2}b_1} - \frac{6a_2^2\sqrt{b_3}C_2}{\sqrt{2b_1}}} + C_1 \int_0^t f(s,z)\,\mathrm{d}s \right)\right\} \\ \times \frac{\varepsilon_2 \sqrt{\frac{b_3}{b_1}} + \varphi^2\left(\mathrm{i}\varepsilon_3 a_2 x \sqrt{\frac{C_2}{2b_1}} - 2\mathrm{i}\varepsilon_1\varepsilon_3 a_2 \sqrt{\frac{-C_2}{2b_1}\left[\sqrt{2}C_1 + \frac{a_2^2 b_2 C_2}{\sqrt{2}b_1} + \frac{6a_2^2\sqrt{b_3}C_2}{\sqrt{2b_1}} \right]} \int_0^t f(s,z)\,\mathrm{d}s \right)}{\varphi\left(\mathrm{i}\varepsilon_3 a_2 x \sqrt{\frac{C_2}{2b_1}} - 2\mathrm{i}\varepsilon_1\varepsilon_3 a_2 \sqrt{\frac{-C_2}{2b_1}\left[\sqrt{2}C_1 + \frac{a_2^2 b_2 C_2}{\sqrt{2}b_1} + \frac{6a_2^2\sqrt{b_3}C_2}{\sqrt{2b_1}} \right]} \int_0^t f(s,z)\,\mathrm{d}s \right)}. \end{aligned} \tag{3.22}$$

(III) For $a_1 = a_3 = 0$ and $b_1 \neq 0$, we have the following four possible solutions:

$$\begin{aligned} u(t,x,z) = a_2 \exp\left\{ \mathrm{i}\left(\varepsilon_1 x \sqrt{\frac{-2b_1 C_1 - a_2^2 b_2 C_2}{2b_1}} + C_1 \int_0^t f(s,z)\,\mathrm{d}s \right)\right\} \\ \times \varphi\left(\mathrm{i}\varepsilon_2 a_2 x \sqrt{\frac{C_2}{2b_1}} - \mathrm{i}\varepsilon_1\varepsilon_2 a_2 \sqrt{\frac{-C_2(2b_1C_1 + a_2^2 b_2 C_2)}{b_1^2}} \int_0^t f(s,z)\,\mathrm{d}s \right). \end{aligned} \tag{3.23}$$

In order to get the exact solutions of (1.2), we suppose that $F(t)$ and $G(t)$ satisfy the following condition:

(A) *Suppose that $F(t)$ and $G(t)$ satisfy that there exist a bounded open set $\mathbf{G} \subset \mathbb{R}_+ \times \mathbb{R}$, $q > 0$ and $r > 0$ such that $u(t,x,z)$, $u_t(t,x,z)$ and $u_{x^2}(t,x,z)$ are uniformly bounded for all $(t,x,z) \in \mathbf{G} \times \mathbb{K}_q(r)$, continuous with respect to $(t,x) \in \mathbf{G}$ for all $z \in \mathbb{K}_q(r)$ and analytic with respect to $z \in \mathbb{K}_q(r)$ for all $(t,x) \in \mathbf{G}$.*

Under condition (A) Theorem 2.1 implies that there exists $U(t,x) \in (\mathscr{S})_{-1}$ such that $u(t,x,z) = \mathscr{H}U(t,x)(z)$ for all $(t,x,z) \in \mathbf{G} \times \mathbb{K}_q(r)$ and that $U(t,x)$ solves (1.2). From the above, we have that $U(t,x)$ is the inverse Hermite transformation of $u(t,x,z)$. Hence, (3.20)–(3.23) yield the solutions of (1.2):

(I) (a) For $a_1 = 0$, $a_3 \neq 0$ and $\delta \neq 0$, we have the following eight possible solutions:

$$U(t,x) = \mu \exp^{\diamond}\left\{ \mathrm{i}\left[\varepsilon_2 x\sqrt{\frac{\mu^2(b_2+c_2)-2C_1}{2}} + C_1\int_0^t F(s)\,\mathrm{d}s \right]\right\}$$
$$\times\left\{ \varepsilon_1\sqrt{\frac{-\gamma c_1}{2C_2}}\,\varphi^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{\frac{\mu^2(b_2+c_2)-2C_1}{2}}\int_0^t F(s)\,\mathrm{d}s\right)\right.$$
$$\left. + \varepsilon_3\sqrt{\frac{-c_1}{2C_2}}\,\psi^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{\frac{\mu^2(b_2+c_2)-2C_1}{2}}\int_0^t F(s)\,\mathrm{d}s\right)\right\}. \tag{3.24}$$

(b) For $a_1 = 0$, $a_3 \neq 0$ and $\gamma \neq 0$, we have the following eight possible solutions with $b_2 = c_2$:

$$U(t,x) = \mu \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_2 x\sqrt{\mu^2 b_2 - C_1} + C_1\int_0^t F(s)\,\mathrm{d}s \right)\right\}$$
$$\times\left\{ \varepsilon_1\sqrt{\frac{-\gamma c_1}{2C_2}}\varphi^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{\mu^2 b_2 - C_1}\int_0^t F(s)\,\mathrm{d}s\right)\right.$$
$$\left. + \varepsilon_3\sqrt{\frac{-c_1}{2C_2}}\psi^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{\mu^2 b_2 - C_1}\int_0^t F(s)\,\mathrm{d}s\right)\right\}. \tag{3.25}$$

(II) For $a_3 = 0$, $a_1 a_2 \neq 0$ and $b_1 \neq 0$, we have the following eight possible solutions with $b_2 = c_2$:

$$U(t,x) = a_2 \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_1 x\sqrt{-\sqrt{2}C_1 - \frac{a_2^2 b_2 C_2}{\sqrt{2}b_1} - \frac{6a_2^2\sqrt{b_3}C_2}{\sqrt{2}b_1}} + C_1\int_0^t F(s)\,\mathrm{d}s \right)\right\}$$
$$\times \frac{\varepsilon_2\sqrt{\frac{b_3}{b_1}} + \varphi^{\diamond 2}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2b_1}} - 2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{-C_2}{2b_1}\left[\sqrt{2}C_1 + \frac{a_2^2 b_2 C_2}{\sqrt{2}b_1} + \frac{6a_2^2\sqrt{b_3}C_2}{\sqrt{2}b_1}\right]}\int_0^t F(s)\,\mathrm{d}s\right)}{\varphi^{\diamond}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2b_1}} - 2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{-C_2}{2b_1}\left[\sqrt{2}C_1 + \frac{a_2^2 b_2 C_2}{\sqrt{2}b_1} + \frac{6a_2^2\sqrt{b_3}C_2}{\sqrt{2}b_1}\right]}\int_0^t F(s)\,\mathrm{d}s\right)}. \tag{3.26}$$

(III) For $a_1 = a_3 = 0$ and $b_1 \neq 0$, we have the following four possible solutions:

$$U(t,x) = a_2 \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{-2b_1C_1 - a_2^2 b_2 C_2}{2b_1}} + C_1\int_0^t F(s)\,\mathrm{d}s \right)\right\}$$
$$\times \varphi^{\diamond}\left(\mathrm{i}\varepsilon_2 a_2 x\sqrt{\frac{C_2}{2b_1}} - \mathrm{i}\varepsilon_1\varepsilon_2 a_2\sqrt{\frac{-C_2(2b_1C_1 + a_2^2 b_2 C_2)}{b_1^2}}\int_0^t F(s)\,\mathrm{d}s\right). \tag{3.27}$$

4. Periodic-like wave solutions of (1.2)

In this section we will give periodic-like wave solutions expressed by various Jacobi elliptic functions for Eq. (1.2) in different values of δ, γ, b_j, c_j, $(j = 1, 2, 3)$. For Eqs. (3.11) and (3.12), Zhou et al. gave various Jacobi elliptic functions solutions in [29]. Here, we only show some examples. For the property of Jacobi elliptic functions, readers can see Wang and Guo's book [21]. Choose $0 < m < 1$ in follows.

(I) For $a_1 = 0$, $a_3 \neq 0$ and $\delta \neq 0$, using relation (3.13), we consider following cases:

Case i: For $b_1=m^2$, $b_2=-1-m^2$ and $b_3=1$, Eq. (3.11) has a solution $\varphi(\xi)=\mathrm{sn}(\xi)$. Choose $c_i=b_i$, $(i=1,2,3)$, $\gamma=0$ and $\delta=1$, we have the following solutions of (1.2):

$$U(t,x)=\frac{m\varepsilon_3\mu}{\sqrt{-2C_2}}\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_2 x\sqrt{-\mu^2(1+m^2)-C_1}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\,\mathrm{sn}^{\diamond}\left(\mu x-2\varepsilon_2\mu\sqrt{-\mu^2(1+m^2)-C_1}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.1}$$

In the limit case when $m\to 1$, $\mathrm{sn}\,\xi\to\tanh\xi$. Eq. (4.1) becomes

$$U(t,x)=\frac{\varepsilon_3\mu}{\sqrt{-2C_2}}\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_2 x\sqrt{-2\mu^2-C_1}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\,\tanh^{\diamond}\left(\mu x-2\varepsilon_2\mu\sqrt{-2\mu^2-C_1}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.2}$$

Case ii: For $b_1=m^2$, $b_2=-1-m^2$, $b_3=1$, $c_1=-m^2$, $c_2=2m^2-1$, $c_3=1-m^2$, or $b_1=-m^2$, $b_2=2m^2-1$, $b_3=1-m^2$, $c_1=m^2$, $c_2=-1-m^2$, $c_3=1$, $\gamma=-1$ and $\delta=1$, we have the solution of (3.12) being $\mathrm{cn}(\xi)$ and the following eight possible solutions of (1.2):

$$U(t,x)=m\mu\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_2 x\sqrt{\frac{\mu^2(m^2-2)-2C_1}{2}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\left\{\frac{\varepsilon_1}{\sqrt{-2C_2}}\mathrm{sn}^{\diamond}\left(\mu x-2\varepsilon_2\mu\sqrt{\frac{\mu^2(m^2-2)-2C_1}{2}}\int_0^t F(s)\,\mathrm{d}s\right)\right.$$
$$\left.+\,\frac{\varepsilon_3}{\sqrt{2C_2}}\mathrm{cn}^{\diamond}\left(\mu x-2\varepsilon_2\mu\sqrt{\frac{\mu^2(m^2-2)-2C_1}{2}}\int_0^t F(s)\,\mathrm{d}s\right)\right\}. \tag{4.3}$$

In the limit case when $m\to 1$, $\mathrm{cn}\,\xi\to\mathrm{sech}\,\xi$. Eq. (4.3) becomes

$$U(t,x)=\mu\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_2 x\sqrt{\frac{-\mu^2-2C_1}{2}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\left\{\frac{\varepsilon_1}{\sqrt{-2C_2}}\tanh^{\diamond}\left(\mu x-2\varepsilon_2\mu\sqrt{\frac{-\mu^2-2C_1}{2}}\int_0^t F(s)\,\mathrm{d}s\right)\right.$$
$$\left.+\,\frac{\varepsilon_3}{\sqrt{2C_2}}\mathrm{sech}^{\diamond}\left(\mu x-2\varepsilon_2\mu\sqrt{\frac{-\mu^2-2C_1}{2}}\int_0^t F(s)\,\mathrm{d}s\right)\right\}. \tag{4.4}$$

Case iii: For $b_1=c_1=m^2$, $b_2=c_2=-(1+m^2)$, $b_3=c_3=1$, $\gamma=1$ and $\delta=0$, we have the following solutions for (1.2):

$$U(t,x)=\frac{m\mu(\varepsilon_1+\varepsilon_3)}{\sqrt{-2C_2}}\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_2 x\sqrt{-\mu^2(1+m^2)-C_1}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\,\mathrm{sn}^{\diamond}\left(\mu x-2\varepsilon_2\mu\sqrt{-\mu^2(1+m^2)-C_1}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.5}$$

In the limit case when $m \to 1$, (4.5) becomes

$$U(t,x) = \frac{\mu(\varepsilon_1+\varepsilon_3)}{\sqrt{-2C_2}} \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_2 x\sqrt{-2\mu^2 - C_1} + C_1 \int_0^t F(s)\,\mathrm{d}s \right)\right\}$$
$$\times \tanh^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{-2\mu^2 - C_1} \int_0^t F(s)\,\mathrm{d}s \right). \tag{4.6}$$

Case iv: For $b_1 = c_1 = -m^2$, $b_2 = c_2 = 2m^2 - 1$, $b_3 = c_3 = 1 - m^2$, $\gamma = 1$ and $\delta = 0$, we have the following solutions for (1.2):

$$U(t,x) = \frac{m\mu(\varepsilon_1+\varepsilon_3)}{\sqrt{2C_2}} \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_2 x\sqrt{\mu^2(2m^2-1) - C_1} + C_1 \int_0^t F(s)\,\mathrm{d}s \right)\right\}$$
$$\times \mathrm{cn}^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{\mu^2(2m^2-1) - C_1} \int_0^t F(s)\,\mathrm{d}s \right). \tag{4.7}$$

In the limit case when $m \to 1$, (4.7) becomes

$$U(t,x) = \frac{\mu(\varepsilon_1+\varepsilon_3)}{\sqrt{2C_2}} \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_2 x\sqrt{\mu^2 - C_1} + C_1 \int_0^t F(s)\,\mathrm{d}s \right)\right\}$$
$$\times \mathrm{sech}^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{\mu^2 - C_1} \int_0^t F(s)\,\mathrm{d}s \right). \tag{4.8}$$

Case v: For $b_1 = c_1 = -1$, $b_2 = c_2 = 2 - m^2$, $b_3 = c_3 = m^2 - 1$, $\gamma = 1$ and $\delta = 0$, Eq. (3.11) has a solution $\varphi(\xi) = \mathrm{dn}\,\xi$. Hence, we have the following possible solutions for (1.2):

$$U(t,x) = \frac{\mu(\varepsilon_1+\varepsilon_3)}{\sqrt{2C_2}} \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_2 x\sqrt{\mu^2(2-m^2) - C_1} + C_1 \int_0^t F(s)\,\mathrm{d}s \right)\right\}$$
$$\times \mathrm{dn}^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{\mu^2(2-m^2) - C_1} \int_0^t F(s)\,\mathrm{d}s \right). \tag{4.9}$$

In the limit case when $m \to 1$, $\mathrm{dn}\,\xi \to \mathrm{cn}\,\xi$. Eq. (4.9) becomes

$$U(t,x) = \frac{\mu(\varepsilon_1+\varepsilon_3)}{\sqrt{2C_2}} \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_2 x\sqrt{\mu^2 - C_1} + C_1 \int_0^t F(s)\,\mathrm{d}s \right)\right\}$$
$$\times \mathrm{cn}^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{\mu^2 - C_1} \int_0^t F(s)\,\mathrm{d}s \right). \tag{4.10}$$

Case vi: For $b_1 = c_1 = 1$, $b_2 = c_2 = -(1 + m^2)$, $b_3 = c_3 = m^2$, $\gamma = 1$ and $\delta = 0$, (3.11) has a solution $\varphi(\xi) = \mathrm{dc}\,\xi$. we have the following solutions for (1.2):

$$U(t,x) = \frac{\mu(\varepsilon_1+\varepsilon_3)}{\sqrt{-2C_2}} \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_2 x\sqrt{-\mu^2(1+m^2) - C_1} + C_1 \int_0^t F(s)\,\mathrm{d}s \right)\right\}$$
$$\times \mathrm{dc}^{\diamond}\left(\mu x - 2\varepsilon_2\mu\sqrt{-\mu^2(1+m^2) - C_1} \int_0^t F(s)\,\mathrm{d}s \right). \tag{4.11}$$

In the limit case when $m \to 0$, $\mathrm{dc}\,\xi \to \sec\xi$. Eq. (4.11) becomes

$$U(t,x)=\frac{\mu(\varepsilon_1+\varepsilon_3)}{\sqrt{-2C_2}}\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_2 x\sqrt{-\mu^2-C_1}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\sec^{\diamond}\left(\mu x-2\varepsilon_2\mu\sqrt{-\mu^2-C_1}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.12}$$

(II) For $a_3=0$, $a_1a_2\neq 0$ and $b_1\neq 0$, we consider following cases:

Case vii: For $b_1=m^2$, $b_2=-(1+m^2)$, $b_3=1$, we have the following eight possible solutions for (1.2):

$$U(t,x)=a_2\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{(m^2-6m+1)a_2^2C_2-2m^2C_1}{\sqrt{2}m^2}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\frac{\frac{\varepsilon_2}{m}+\mathrm{sn}^{\diamond 2}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2m^2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2[(m^2-6m+1)a_2^2C_2-2m^2C_1]}{2\sqrt{2}m^4}}\int_0^t F(s)\,\mathrm{d}s\right)}{\mathrm{sn}^{\diamond}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2m^2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2[(m^2-6m+1)a_2^2C_2-2m^2C_1]}{2\sqrt{2}m^4}}\int_0^t F(s)\,\mathrm{d}s\right)}. \tag{4.13}$$

In the limit case when $m\to 1$, (4.13) becomes

$$U(t,x)=a_2\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{-\sqrt{2}(2a_2^2C_2+C_1)}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\frac{\varepsilon_2+\tanh^{\diamond 2}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{-C_2[2a_2^2C_2+C_1]}{\sqrt{2}}}\int_0^t F(s)\,\mathrm{d}s\right)}{\tanh^{\diamond}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{-C_2[2a_2^2C_2+C_1]}{\sqrt{2}}}\int_0^t F(s)\,\mathrm{d}s\right)}. \tag{4.14}$$

Case viii: For $b_1=-m^2$, $b_2=2m^2-1$, $b_3=1-m^2$, we have the following eight possible solutions for (1.2):

$$U(t,x)=a_2\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{(2m^2-1+6\mathrm{i}m\sqrt{1-m^2})a_2^2C_2-2m^2C_1}{\sqrt{2}m^2}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\frac{\varepsilon_2\frac{\sqrt{m^2-1}}{m}+\mathrm{cn}^{\diamond 2}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{-C_2}{2m^2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2[2m^2C_1-(2m^2-1+6\mathrm{i}m\sqrt{1-m^2})a_2^2C_2]}{2\sqrt{2}m^4}}\int_0^t F(s)\,\mathrm{d}s\right)}{\mathrm{cn}^{\diamond}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{-C_2}{2m^2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2[2m^2C_1-(2m^2-1+6\mathrm{i}m\sqrt{1-m^2})a_2^2C_2]}{2\sqrt{2}m^4}}\int_0^t F(s)\,\mathrm{d}s\right)}. \tag{4.15}$$

In the limit case when $m\to 1$, (4.15) becomes

$$U(t,x)=a_2\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2-2C_1}{\sqrt{2}}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\mathrm{sech}^{\diamond}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{-C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2[2C_1-a_2^2C_2]}{2\sqrt{2}}}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.16}$$

Case ix: For $b_1=-1$, $b_2=2-m^2$ and $b_3=m^2-1$, we have the following eight possible solutions for (1.2):

$$U(t,x)=a_2\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{(2-m^2-6\sqrt{1-m^2})a_2^2C_2-2C_1}{\sqrt{2}}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\frac{\varepsilon_2\sqrt{1-m^2}+\mathrm{dn}^{\diamond 2}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{-C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2[2C_1-(2-m^2-6\sqrt{1-m^2})a_2^2C_2]}{2\sqrt{2}}}\int_0^t F(s)\,\mathrm{d}s\right)}{\mathrm{dn}^{\diamond}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{-C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2[2C_1-(2-m^2-6\sqrt{1-m^2})a_2^2C_2]}{2\sqrt{2}}}\int_0^t F(s)\,\mathrm{d}s\right)}. \tag{4.17}$$

In the limit case when $m\to 1$, (4.17) becomes

$$U(t,x)=a_2\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2-2C_1}{\sqrt{2}}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\,\mathrm{cn}^{\diamond}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{-C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2(2C_1-a_2^2C_2)}{2\sqrt{2}}}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.18}$$

Case x: For $b_1=1$, $b_2=-(1+m^2)$ and $b_3=m^2$, we have the following eight possible solutions for (1.2):

$$U(t,x)=a_2\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{(m^2-6m+1)a_2^2C_2-2C_1}{\sqrt{2}}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\frac{m\varepsilon_2+\mathrm{dc}^{\diamond 2}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2[(m^2-6m+1)a_2^2C_2-2C_1]}{2\sqrt{2}}}\int_0^t F(s)\,\mathrm{d}s\right)}{\mathrm{dc}^{\diamond}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2\left[(m^2-6m+1)a_2^2C_2-2C_1\right]}{2\sqrt{2}}}\int_0^t F(s)\,\mathrm{d}s\right)}. \tag{4.19}$$

In the limit case when $m\to 0$, (4.19) becomes

$$U(t,x)=a_2\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2-2C_1}{\sqrt{2}}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\,\sec^{\diamond}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2(a_2^2C_2-2C_1)}{2\sqrt{2}}}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.20}$$

(III) For $a_1=a_3=0$ and $b_1\neq 0$, we consider following cases:
Case xi: For $b_1=m^2$, $b_2=-(1+m^2)$, $b_3=1$, we have the following four possible solutions for (1.2):

$$U(t,x)=a_2\exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{(1+m^2)a_2^2C_2-2m^2C_1}{2m^2}}+C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times\,\mathrm{sn}^{\diamond}\left(\mathrm{i}\varepsilon_2 a_2 x\frac{\sqrt{2C_2}}{2m}-\mathrm{i}\varepsilon_1\varepsilon_2 a_2\frac{\sqrt{C_2[(1+m^2)a_2^2C_2-2m^2C]}}{m^2}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.21}$$

In the limit case when $m \to 1$, (4.21) becomes

$$U(t,x) = a_2 \exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{a_2^2C_2 - C_1} + C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times \tanh^{\diamond}\left(\mathrm{i}\varepsilon_2 a_2 x\frac{\sqrt{2C_2}}{2} - \mathrm{i}\varepsilon_1\varepsilon_2 a_2\sqrt{2C_2(a_2^2C_2 - C_1)}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.22}$$

Case xii: For $b_1 = -m^2$, $b_2 = 2m^2 - 1$, $b_3 = 1 - m^2$, we have the following four possible solutions for (1.2):

$$U(t,x) = a_2 \exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{(2m^2-1)a_2^2C_2 - 2m^2C_1}{2m^2}} + C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times \mathrm{cn}^{\diamond}\left(\mathrm{i}\varepsilon_2 a_2 x\frac{\sqrt{-2C_2}}{2m} - \mathrm{i}\varepsilon_1\varepsilon_2 a_2\frac{\sqrt{C_2[2m^2C_1 - (2m^2-1)a_2^2C_2]}}{m^2}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.23}$$

In the limit case when $m \to 1$, (4.23) becomes

$$U(t,x) = a_2 \exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2 - 2C_1}{2}} + C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times \mathrm{sech}^{\diamond}\left(\mathrm{i}\varepsilon_2 a_2 x\sqrt{\frac{-2C_2}{2}} - \mathrm{i}\varepsilon_1\varepsilon_2 a_2\sqrt{C_2(2C_1 - a_2^2C_2)}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.24}$$

Case xiii: For $b_1 = -1$, $b_2 = 2 - m^2$, $b_3 = m^2 - 1$, we have the following four possible solutions for (1.2):

$$U(t,x) = a_2 \exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{(2-m^2)a_2^2C_2 - 2C_1}{2}} + C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times \mathrm{dn}^{\diamond}\left(\mathrm{i}\varepsilon_2 a_2 x\sqrt{\frac{-C_2}{2}} - \mathrm{i}\varepsilon_1\varepsilon_2 a_2\sqrt{C_2(2C_1 - a_2^2C_2(2-m^2))}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.25}$$

In the limit case when $m \to 1$, (4.25) becomes

$$U(t,x) = a_2 \exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2 - 2C_1}{2}} + C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times \mathrm{cn}^{\diamond}\left(\mathrm{i}\varepsilon_2 a_2 x\sqrt{\frac{-C_2}{2}} - \mathrm{i}\varepsilon_1\varepsilon_2 a_2\sqrt{C_2(2C_1 - a_2^2C_2)}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.26}$$

Case xiv: For $b_1 = 1$, $b_2 = -(1+m^2)$ and $b_3 = m^2$, we have the following four possible solutions for (1.2):

$$U(t,x) = a_2 \exp^{\diamond}\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2(1+m^2) - 2C_1}{2}} + C_1\int_0^t F(s)\,\mathrm{d}s\right)\right\}$$
$$\times \mathrm{dc}^{\diamond}\left(\mathrm{i}\varepsilon_2 a_2 x\sqrt{\frac{C_2}{2}} - \mathrm{i}\varepsilon_1\varepsilon_2 a_2\sqrt{C_2[a_2^2C_2(1+m^2) - 2C_1]}\int_0^t F(s)\,\mathrm{d}s\right). \tag{4.27}$$

In the limit case when $m \to 0$, (4.27) becomes

$$U(t,x) = a_2 \exp^{\diamond}\left\{ \mathrm{i}\left(\varepsilon_1 x \sqrt{\frac{a_2^2 C_2 - 2C_1}{2}} + C_1 \int_0^t F(s)\,\mathrm{d}s \right) \right\}$$

$$\times \sec^{\diamond}\left(\mathrm{i}\varepsilon_2 a_2 x \sqrt{\frac{C_2}{2}} - \mathrm{i}\varepsilon_1\varepsilon_2 a_2 \sqrt{C_2(a_2^2 C_2 - 2C_1)} \int_0^t F(s)\,\mathrm{d}s \right). \tag{4.28}$$

Since Wick versions of functions are usually difficult to evaluate, we will give some nonWick versions of solutions of (1.2) in special cases.

Let $F(t) = f(t) + aW(t)$, where a is a constant, $f(t)$ is a integrable or bounded measurable function on $\mathbb{R}_+$, $W(t)$ is Wiener white noise, i.e., $W_t = \dot{B}_t$, B_t is a Brown motion. Using $\exp^{\diamond}\{B_t\} = \exp\{B_t - \frac{1}{2}t^2\}$ and the definitions of $\sec\,\xi$, $\operatorname{sech}\xi$ and $\tanh\,\xi$, we can get solutions of (1.2) as follows:

$$U(t,x) = \frac{\varepsilon_3 \mu}{\sqrt{-2C_2}} \exp\left\{ \mathrm{i}\left(\varepsilon_2 x \sqrt{-2\mu^2 - C_1} + C_1 \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right) \right\}$$

$$\times \tanh\left(\mu x - 2\varepsilon_2 \mu \sqrt{-2\mu^2 - C_1} \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right),$$

$$U(t,x) = \mu \exp\left\{ \mathrm{i}\left(\varepsilon_2 x \sqrt{\frac{-\mu^2 - 2C_1}{2}} + C_1 \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right) \right\}$$

$$\times \left\{ \frac{\varepsilon_1}{\sqrt{-2C_2}} \tanh\left(\mu x - 2\varepsilon_2 \mu \sqrt{\frac{-\mu^2 - 2C_1}{2}} \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right) \right.$$

$$\left. + \frac{\varepsilon_3}{\sqrt{2C_2}} \operatorname{sech}\left(\mu x - 2\varepsilon_2 \mu \sqrt{\frac{-\mu^2 - 2C_1}{2}} \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right) \right\},$$

$$U(t,x) = \frac{\mu(\varepsilon_1 + \varepsilon_3)}{\sqrt{-2C_2}} \exp\left\{ \mathrm{i}\left(\varepsilon_2 x \sqrt{-2\mu^2 - C_1} + C_1 \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right) \right\}$$

$$\times \tanh\left(\mu x - 2\varepsilon_2 \mu \sqrt{-2\mu^2 - C_1} \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right),$$

$$U(t,x) = \frac{\mu(\varepsilon_1 + \varepsilon_3)}{\sqrt{2C_2}} \exp\left\{ \mathrm{i}\left(\varepsilon_2 x \sqrt{\mu^2 - C_1} + C_1 \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right) \right\}$$

$$\times \operatorname{sech}\left(\mu x - 2\varepsilon_2 \mu \sqrt{\mu^2 - C_1} \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right),$$

$$U(t,x) = \frac{\mu(\varepsilon_1 + \varepsilon_3)}{\sqrt{-2C_2}} \exp\left\{ \mathrm{i}\left(\varepsilon_2 x \sqrt{-\mu^2 - C_1} + C_1 \left(a(B_t - t^2/2) + \int_0^t f(s)\,ds \right) \right) \right\}$$

$$\times \sec\left(\mu x - 2\varepsilon_2 \mu \sqrt{-\mu^2 - C_1} \left(a(B_t - t^2/2) + \int_0^t f(s)\,\mathrm{d}s \right) \right),$$

$$U(t,x)=a_2\exp\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{-\sqrt{2}(2a_2^2C_2+C_1)}+C_1\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right)\right\}$$
$$\times\frac{\varepsilon_2+\tanh^2\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{-C_2[2a_2^2C_2+C_1]}{\sqrt{2}}}(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s)\right)}{\tanh\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{-C_2[2a_2^2C_2+C_1]}{\sqrt{2}}}(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s)\right)},$$

$$U(t,x)=a_2\exp\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2-2C_1}{\sqrt{2}}}+C_1\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right)\right\}$$
$$\times\operatorname{sech}\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{-2C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2\left(2C_1-a_2^2C_2\right)}{2\sqrt{2}}}\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right),$$

$$U(t,x)=a_2\exp\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2-2C_1}{\sqrt{2}}}+C_1\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right)\right\}$$
$$\times\sec\left(\mathrm{i}\varepsilon_3 a_2 x\sqrt{\frac{C_2}{2}}-2\mathrm{i}\varepsilon_1\varepsilon_3 a_2\sqrt{\frac{C_2(a_2^2C_2-2C_1)}{2\sqrt{2}}}\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right),$$

$$U(t,x)=a_2\exp\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{a_2^2C_2-C_1}+C_1\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right)\right\}$$
$$\times\tanh\left(\mathrm{i}\varepsilon_2 a_2 x\frac{\sqrt{2C_2}}{2}-\mathrm{i}\varepsilon_1\varepsilon_2 a_2\sqrt{2C_2(a_2^2C_2-C_1)}\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right),$$

$$U(t,x)=a_2\exp\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2-2C_1}{2}}+C_1\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right)\right\}$$
$$\times\operatorname{sech}\left(\mathrm{i}\varepsilon_2 a_2 x\sqrt{\frac{-C_2}{2}}-\mathrm{i}\varepsilon_1\varepsilon_2 a_2\sqrt{C_2(2C_1-a_2^2C_2)}\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right),$$

$$U(t,x)=a_2\exp\left\{\mathrm{i}\left(\varepsilon_1 x\sqrt{\frac{a_2^2C_2-2C_1}{2}}+C_1\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right)\right\}$$
$$\times\sec\left(\mathrm{i}\varepsilon_2 a_2 x\sqrt{\frac{C_2}{2}}-\mathrm{i}\varepsilon_1\varepsilon_2 a_2\sqrt{C_2(a_2^2C_2-2C_1)}\left(a(B_t-t^2/2)+\int_0^t f(s)\,\mathrm{d}s\right)\right).$$

Remark 4.1. (i) Since $\Phi^{\diamond}(x)=\Phi(x)$ for any nonrandom function $\Phi(x)$, Eqs. (3.24)–(3.27) and (4.1)–(4.28) are solutions of the variable coefficient NLS equation (1.1) if $F(t)=f(t)$ and $G(t)=g(t)$ are nonrandom bounded measurable or integrable functions on $\mathbb{R}_+$.

(ii) Since there is a unitary map between the Wiener and the Poisson white noise spaces, we can obtain the solution of the Poissonian SPDE simply applying this map to the solution of the corresponding Gaussian SPDE. A nice, concise account of this connection was given by Benth and Gjerde in [2]. We can see Section 4.9 of [9] also. Hence, we can get solutions as we do in Sections 3 and 4 if the coefficients $F(t)$ and $G(t)$ are Poissonian white noise functionals in (1.2).

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